2D H - atom in an arbitrary magnetic field via pseudoperturbation expansions through the quantum number l

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Abstract

The pseudoperturbative shifted - l expansion technique (PSLET) is introduced to determine nodeless states of the 2D Schrödinger equation with arbitrary cylindrically symmetric potentials. Exact energy eigenvalues and eigenfunctions for the 2D Coulomb and harmonic oscillator potentials are reproduce. Moreover, exact energy eigenvalues, compared to those obtained by numerical solution [11], were obtained for the hybrid of the 2D Coulomb and oscillator potentials.

1 Introduction

Recent advances in nanofabrication technology have made possible to create low dimensional structures [1-4 and references therein] such as quantum wells, wires and dots. In almost most of problems concerning such structures the eigenspectrum of the 2D- Schrődinger equation is investigated. For example, the 2D hydrogenic energy levels in a constant magnetic field of arbitrary strength has been a subject of numerous theoretical and experimental investigations [1-14].

Taut [6] has found analytical solutions for this problem. However, no solutions for nodeless states were found. Martin et al [10] have used a two - point quasifractional approximation and found excellent interpolation between the weak and strong magnetic field perturbation expansions. MacDonald and Ritchie [8] have however used a two - point Padé approximation. The curves obtained from different Padé approximates are very different. No regular pattern appears and the results become unreliable. More detailed discussion can be found in ref [10]. Zhu et al [9] have used a power series expansion method. Their results compared excellently with the direct numerical integration [12] and perturbation [8] methods in the weak field regime. However, neither Zhu's nor direct numerical integration results seem to approach the perturbation results at the strong field limit. Villalba and Pino [11] have used a finite - difference scheme with a linear mesh of up to 2000 points and failed to provide a good estimation of the ground state for the 2D hydrogen atom. On the other hand, their variational solution obtained using hydrogen basis is in good agreement with that of Martin et al [10] at the weak field limit. Using the oscillator basis, their results were in good agreement at the

strong field limit. Their solutions could not therefore provide information about how the energy shifts in the intermediate range of the magnetic field. Mustafa [5] and Quiroga et al [7] have used the shifted N - expansion technique (SLNT) and their results were in good agreements with those of Martin et al [10].

In this work, section two considers the 2D Shcrődinger equation with an arbitrary cylindrically symmetric potential. By introducing a technique, which we will call Pseudoperturbative Shifted - l Expansion Technique (PSLET), we obtain analytical expressions for both eigenenergies and eigenfunctions for the 2D Schrődinger equation. In section 3, we test our method for both 2D Coulomb and harmonic oscillator potentials, and the results found to be exact. We also consider the 2D hydrogenic energy levels in a constant magnetic field of arbitrary strength. PSLET results are shown to compare excellently with the perturbation [8], direct numerical integration [12], and series expansion [9] methods. They are also shown to be in exact agreements with the numerical solution of the 2D Schrödinger equation [10]. We conclude with remarkes in section 4.

2 Theory of 2D-PSLET

The Schrödinger equation for an arbitrary cylindrically symmetric potential $V(\rho)$ (in $\hbar = 2m = 1$ units) is

$$\left[-\frac{d^2}{d\rho^2} + \frac{4l^2-1}{4\rho^2} + V(\rho)\right]\Psi(\rho) = E\Psi(\rho), \tag{1}$$

where $\rho^2 = x^2 + y^2$, $l = |m|$ and m is the magnetic quantum number. If l is shifted through the relation $\bar{l} = l - \beta$, Eq.(1) becomes

$$\left\{-\frac{d^2}{d\rho^2}+\tilde{V}(\rho)\right\}\Psi(\rho)=E\Psi(\rho), \tag{2}$$

with

$$\tilde{V}(\rho)=\frac{(\bar{l}+\beta+1/2)(\bar{l}+\beta-1/2)}{\rho^2}+\frac{\bar{l}^2}{Q}V(\rho). \tag{3}$$

Where Q is a scale to be determined below and set equal to $\bar{l}^2$ at the end of the calculations, and β is a suitable shift to be determined and is introduced partly to avoid the trivial case $l=0$.

The systematic procedure of PSLET starts with making use of Taylor's theorem and expanding (2) about an arbitrary point (for now) on the ρ - axis. It is convenient then to transform the coordinates in (2) via the relation

$$x=\bar{l}^{1/2}(\rho-\rho_o)/\rho_o, \tag{4}$$

where ρ_o is our arbitrary point. Expansions about $x=0$ yield

$$\left[-\frac{d^2}{dx^2}+\tilde{V}(x(\rho))\right]\Psi(x)=\frac{\rho_o^2}{\bar{l}}E\Psi(x), \tag{5}$$

where

$$\begin{aligned}\tilde{V}(x(\rho)) &= \rho_o^2\bar{l}\left[\frac{1}{\rho_o^2}+\frac{V(\rho_o)}{Q}\right]+\bar{l}^{1/2}\left[-2x+\frac{V^{'}(\rho_o)\rho_o^3x}{Q}\right]\\ &+ \left[3x^2+\frac{V^{''}(\rho_o)\rho_o^4x^2}{2Q}\right]+2\beta\sum_{n=1}^{\infty}(-1)^n(n+1)x^n\bar{l}^{-n/2}\\ &+ \sum_{n=3}^{\infty}\left[(-1)^n(n+1)x^n+\left(\frac{d^nV(\rho_o)}{d\rho_o^n}\right)\frac{\rho_o^2(\rho_ox)^n}{n!Q}\right]\bar{l}^{-(n-2)/2}\end{aligned}$$

$$+ \quad (\beta^2 - 1/4) \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \bar{l}^{-(n+2)/2} + 2\beta. \tag{6}$$

It is convenient to expand E as

$$E = \sum_{n=-2}^{\infty} E^{(n)} \bar{l}^{-n}. \tag{7}$$

where the prime of $V(\rho_o)$ denotes derivative with respect to ρ_o. Equation (6) when compared to Schrödinger equation for one - dimensional anharmonic oscillator

$$\left[-\frac{d^2}{dx^2} + \frac{1}{4} w^2 x^2 + \varepsilon_o + P(x) \right] X_{n_\rho}(x) = \lambda_{n_\rho} X_{n_\rho}(x), \tag{8}$$

where $P(x)$ is a perturbation term and ε_o is a constant, implies

$$\varepsilon_o = \bar{l} \left[1 + \frac{\rho_o^2 V(\rho_o)}{Q} \right] + 2\beta + \frac{(\beta^2 - 1/4)}{\bar{l}}, \tag{9}$$

$$\lambda_{n_\rho} = \varepsilon_o + (n_\rho + 1/2) w + \lambda^{(0)} / \bar{l} + \sum_{n=2}^{\infty} \lambda^{(n-1)} \bar{l}^{-n}, \tag{10}$$

and

$$\lambda_{n_\rho} = \rho_o^2 \sum_{n=-2}^{\infty} E^{(n)} \bar{l}^{-(n+1)}. \tag{11}$$

Equations (10) and (11) yield

$$E^{(-2)} = \frac{1}{\rho_o^2} + \frac{V(\rho_o)}{Q} \tag{12}$$

$$E^{(-1)} = \frac{1}{\rho_o^2}\left[2\beta + (n_\rho + 1/2)w\right] \tag{13}$$

$$E^{(0)} = \frac{1}{\rho_o^2}\left[\beta^2 - 1/4 + \lambda^{(0)}\right] \tag{14}$$

$$E^{(n)} = \lambda^{(n)}/\rho_o^2 \;\; ; \quad n \geq 1. \tag{15}$$

Here ρ_o is chosen to minimize $E^{(-2)}$, i. e.

$$\frac{dE^{(-2)}}{d\rho_o} = 0 \quad and \quad \frac{d^2E^{(-2)}}{d\rho_o^2} > 0, \tag{16}$$

which in turn gives, with $\bar{l} = \sqrt{Q}$,

$$l - \beta = \sqrt{\frac{\rho_o^3 V'(\rho_o)}{2}}. \tag{17}$$

The shifting parameter β is determined by choosing the next leading correction to the energy eigenvalue, $E^{(-1)}$, to vanish. This choice is physically motivated by requiring the agreement between 2D-PSLET eigenvalues and eigenfunctions and the exact analytical eigenvalues and eigenfunctions for both the Coulomb and harmonic oscillator potentials. Hence

$$\beta = -\frac{1}{2}\left[(n_\rho + \frac{1}{2})w\right], \tag{18}$$

where

$$w = 2\sqrt{3 + \frac{\rho_o V''(\rho_o)}{V'(\rho_o)}}. \tag{19}$$

Then equation (6) becomes

$$\tilde{V}(x(\rho)) = \rho_o^2 \bar{l} E^{(-2)} + \sum_{n=0}^{\infty} v^{(n)}(x) \bar{l}^{-n/2}, \tag{20}$$

where

$$v^{(0)}(x) = \frac{1}{4} w^2 x^2 + 2\beta, \tag{21}$$

$$v^{(1)}(x) = -4\beta x - 4x^3 + \frac{\rho_o^5 V'''(\rho_o)}{6Q} x^3, \tag{22}$$

$$v^{(2)}(x) = (\beta^2 - 1/4) + 6\beta x^2 + \left(5 + \frac{\rho_o^6 V''''(\rho_o)}{24Q}\right) x^4, \tag{23}$$

and for $n \geq 3$

$$\begin{aligned} v^{(n)}(x) &= (-1)^n 2\beta(n+1)x^n + (-1)^n(\beta^2 - 1/4)(n-1)x^{n-2} \\ &+ \left[(-1)^n(n+3) + \frac{\rho_o^{n+4}}{Q(n+2)!} \frac{d^{n+2}V(\rho_o)}{d\rho_o^{n+2}}\right] x^{n+2}. \end{aligned} \tag{24}$$

Equation (5) thus becomes

$$\left[-\frac{d^2}{dx^2}+\sum_{n=0}^{\infty}v^{(n)}\bar{l}^{-n/2}\right]\Psi_{n_\rho}(x)= \\ \left[\frac{1}{\bar{l}}\left(\beta^2-\frac{1}{4}+\lambda^{(0)}\right)+\sum_{n=2}^{\infty}\lambda^{(n-1)}\bar{l}^{-n}\right]\Psi_{n_\rho}(x). \tag{25}$$

For nodeless unnormalized wave functions $n_\rho = 0$ and

$$\Psi_0(x(\rho)) = exp(U(x)), \tag{26}$$

which when substituted in equation (25) yields

$$-\left[U^{''}(x)+U^{'}(x)U^{'}(x)\right]+\sum_{n=0}^{\infty}v^{(n)}(x)\bar{l}^{-n/2} = \frac{1}{\bar{l}}\left(\beta^2-\frac{1}{4}+\lambda^{(0)}\right) \\ +\sum_{n=2}^{\infty}\lambda^{(n-1)}\bar{l}^{-n}, \tag{27}$$

where prime of $U(x)$ denotes derivative with respect to x. It is evident that this equation admits solution of the form

$$U^{'}(x)=\sum_{n=0}^{\infty}U^{(n)}(x)\bar{l}^{-n/2}+\sum_{n=0}^{\infty}G^{(n)}(x)\bar{l}^{-(n+1)/2}, \tag{28}$$

where

$$U^{(n)}(x)=\sum_{j=0}^{n+1}D_{j,n}x^{2j-1} \quad ; \quad D_{0,n}=0, \tag{29}$$

$$G^{(n)}(x) = \sum_{j=0}^{n+1} C_{j,n} x^{2j}. \tag{30}$$

Substituting equations (28) into equation (27) implies

$$\begin{aligned} - & \sum_{n=0}^{\infty} \left[U^{(n)'} \bar{l}^{-n/2} + G^{(n)'} \bar{l}^{-(n+1)/2} \right] \\ - & \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left[U^{(n)} U^{(p)} \bar{l}^{-(n+p)/2} + G^{(n)} G^{(p)} \bar{l}^{-(n+p+2)/2} + 2U^{(n)} G^{(p)} \bar{l}^{-(n+p+1)/2} \right] \\ + & \sum_{n=0}^{\infty} v^{(n)} \bar{l}^{-n/2} = \frac{1}{\bar{l}} \left(\beta^2 - \frac{1}{4} + \lambda^{(0)} \right) + \sum_{n=2}^{\infty} \lambda^{(n-1)} \bar{l}^{-n}, \end{aligned} \tag{31}$$

where primes of $U^{(n)}(x)$ and $G^{(n)}(x)$ denote derivatives with respect to x. Equating terms of same order in $\bar{l}$ one obtains

$$- [U^{(0)'} + U^{(0)} U^{(0)}] + v^{(0)} = 0, \tag{32}$$

$$U^{(0)'}(x) = D_{1,0} \qquad ; \quad D_{1,0} = -\omega/2 \tag{33}$$

Integration over dx yields

$$U^{(0)}(x) = -\omega x/2 \tag{34}$$

Similarly,

$$- [U^{(1)'} + G^{(0)'}] - 2U^{(0)} U^{(1)} - 2U^{(0)} G^{(0)} + v^{(1)} = 0, \tag{35}$$

$$U^{(1)}(x) = 0, \tag{36}$$

$$G^{(0)}(x) = C_{0,0} + C_{1,0}x^2, \tag{37}$$

$$C_{1,0} = -\frac{B_1}{w}, \tag{38}$$

$$C_{0,0} = \frac{1}{w}(2C_{1,0} - w), \tag{39}$$

$$B_1 = -4 + \frac{\rho_o^5}{6Q}\frac{d^3}{d\rho_o^3}V(\rho_o), \tag{40}$$

$$-[U^{(2)'} + G^{(1)'}] - \sum_{n=0}^{2} U^{(n)}U^{(2-n)} - G^{(0)}G^{(0)}$$
$$-2\sum_{n=0}^{1} U^{(n)}G^{(1-n)} + v^{(2)} = \beta^2 - \frac{1}{4} + \lambda^{(0)}, \tag{41}$$

$$U^{(2)}(x) = D_{1,2}x + D_{2,2}x^3, \tag{42}$$

$$G^{(1)}(x) = 0, \tag{43}$$

$$D_{2,2} = \frac{1}{w}(C_{1,0}^2 - B_2) \tag{44}$$

$$D_{1,2} = \frac{1}{w}(3D_{2,2} + 2C_{0,0}C_{1,0} - 6\beta), \tag{45}$$

$$B_2 = 5 + \frac{\rho_o^6}{24Q}\frac{d^4}{d\rho_o^4}V(\rho_o), \tag{46}$$

$$\lambda^{(0)} = -(D_{1,2} + C_{0,0}^2). \tag{47}$$

... and so on. Clearly one can calculate the energy eigenvalues and the eigenfunctions from the knowledge of $\mathrm{C}_{n,m}$ and $\mathrm{D}_{n,m}$ in a hierarchical manner. However, it is for the convenience of this study to conclude the procedure here and give the energy eigenvalues and eigenfunctions as

$$E = E^{(-2)}\bar{l}^2 + E^{(0)} + E^{(1)}\bar{l}^{-1} + E^{(2)}\bar{l}^{-2} + O(\bar{l}^{-3}), \tag{48}$$

$$\begin{aligned} U^{'}(x(\rho)) &= U^{(0)} + [G^{(0)} + U^{(1)}]\bar{l}^{-1/2} + [G^{(1)} + U^{(2)}]\bar{l}^{-1} \\ &\quad + [G^{(2)} + U^{(3)}]\bar{l}^{-3/2} + [G^{(3)} + U^{(4)}]\bar{l}^{-2} + [G^{(4)} + U^{(5)}]\bar{l}^{-5/2} \\ &\quad + [G^{(5)} + U^{(6)}]\bar{l}^{-3} + O(\bar{l}^{-7/2}). \end{aligned} \tag{49}$$

From which we find

$$\Psi_0(x(\rho)) = exp\left(\int U^{'}(x)dx\right). \tag{50}$$

3 Applications, results and discussions

In this section the above analytical expressions of 2D - PSLET are investigated through the 2D Coulomb, harmonic oscillator and the hybrid of the

two potentials.

3.1 The Coulomb potential $V(\rho) = -2/\rho$

Following 2D - PSLET theory one obtains, for nodeless states,

$$E^{(-2)}\bar{l}^2 = -\frac{1}{\bar{l}^2} \;\; ; \;\; \bar{l} = |m| + 1/2 \;\; , \;\; \rho_o = \bar{l}^2, \tag{51}$$

$$E^{(0)} = E^{(1)}\bar{l}^{-1} = E^{(2)}\bar{l}^{-2} = \cdots = 0, \tag{52}$$

$$U(x) = -\bar{l}y + \bar{l}\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \frac{y^6}{6} + \frac{y^7}{7} - \frac{y^8}{8}\right), \tag{53}$$

where $y = x\bar{l}^{-1/2}$. It is obvious that the second term in equation (85) is the infinite geometric series expansion for $ln(1+y)^{\bar{l}}$. Equation (85) can thus be approximated by

$$U(x) \simeq -\bar{l}y + ln(1+y)^{\bar{l}}, \tag{54}$$

which in turn implies that

$$\Psi_{0,m}(\rho) = \left(\frac{\rho}{\rho_o}\right)^{\bar{l}} e^{\bar{l}} e^{-\bar{l}\rho/\rho_o}. \tag{55}$$

Hence the nodeless radial parts of the wave functions are

$$R_{0,m}(\rho) = N\rho^{\bar{l}-1} e^{-\bar{l}\rho/\rho_o}, \tag{56}$$

where N is the normalization constant given by the relation

$$N = \frac{(2\bar{l}/\rho_o)^{\bar{l}+1/2}}{\sqrt{(2\bar{l})!}}. \tag{57}$$

Equation (88) evidently gives the exact expressions for the normalized nodeless radial parts of the wave functions.

Finally, the exact energy eigenvalues for any nodeless orbital state are

$$E_{0,m} = -(|m| + 1/2)^{-2}. \tag{58}$$

3.2 The harmonic oscillator potential $V(\rho) = \gamma^2\rho^2/4$

For this potential, 2D - PSLET procedure yields, for nodeless states,

$$E^{(-2)}\bar{l}^2 = \gamma\bar{l} \quad ; \quad \bar{l} = |m| + 1, \tag{59}$$

$$E^{(0)} = E^{(1)}\bar{l}^{-1} = E^{(2)}\bar{l}^{-2} = \cdots = 0, \tag{60}$$

$$\begin{aligned} U(x) &= -\frac{1}{2}\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \frac{y^6}{6}\right) \\ &\quad +\bar{l}\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \frac{y^6}{6} + \frac{y^7}{7} - \frac{y^8}{8}\right) \\ &\quad -\bar{l}\frac{y^2}{2} - \bar{l}y, \end{aligned} \tag{61}$$

Obviously the terms in brackets are the infinite geometric series expansions for $ln(1+y)$. Equation (93) can thus be approximated by

$$U(x) \simeq ln(1+y)^{-1/2} + ln(1+y)^{\bar{l}} - \bar{l}y - \bar{l}\frac{y^2}{2}, \tag{62}$$

which in turn implies that

$$\Psi_{0,m}(\rho) = \left(\frac{\rho}{\rho_o}\right)^{\bar{l}-1/2} e^{-\gamma\rho^2/4} \quad ; \quad \rho_o = \sqrt{2\bar{l}/\gamma}. \tag{63}$$

Hence, the nodeless radial parts of the wave functions are

$$R_{0,m}(\rho) = N\rho^{\bar{l}-3/2}e^{-\gamma\rho^2/4}, \tag{64}$$

where the normalization constant N is given through the relation

$$N^2 = \frac{2^{\bar{l}+1/2}(\gamma/2)^{\bar{l}-1/2}}{1\cdot 3\cdot 5\cdot\cdots\cdot(2\bar{l}-2)}\sqrt{\frac{\gamma}{2\pi}}. \tag{65}$$

Equation (96) clearly yields the exact expressions for the normalized nodeless radial parts of the wave functions. The exact energy eigenvalues for any nodeless state are given by

$$E_{0,m} = \gamma(|m|+1). \tag{66}$$

3.3 A hybrid of Coulomb and oscillator potentials

Perhaps the most interesting form of such a hybrid model is

$$V(\rho) = m\gamma - \frac{2}{\rho} + \frac{\gamma^2\rho^2}{4}, \tag{67}$$

where m is the magnetic quantum number. This potential (99) describes a 2D electron gas in the x-y plane in the presence of a hydrogenic potential, representing the interaction between a conduction electron and a donor impurity, in a magnetic field in the z - direction. It also resembles the potential model that describes a donor impurity in a single quantum dot moving in a magnetic field in the z - direction [4-14].

It is obvious from equation (17), along with (18) and (19), that one can hardly find an analytical solution for ρ_o, in terms of γ and m, using the potential in (99). One has therefore to appeal to numerical techniques to solve for ρ_o for each γ and m. Once ρ_o is found, the energy eigenvalues as well as eigenfunctions can be obtained through the 2D-PSLET theory described in section 2.

In tables 1-4 we list the energy eigenvalues in such a way that the contribution of each energy correction is made clear. The tables show that 2D-PSLET results are rapidly convergent. It is also evident that the energy eigenvalues for states with larger $|m|$ converge more rapidly than those with smaller $|m|$. Such a tendency was obvious from the very moment of the invention of the deceptive perturbation parameter $1/\bar{l}$, $\bar{l} = |m| - \beta$ and β is always negative.

It could be interesting to know that the computation time for all ρ_o's needed for the entries in tables 1-4 is less than 30 sec, and for each entry is at most 20 sec, including the eigenfunctions for each of them. ρ_o is computed using EUREKA and the eigenvalues as well as eigenfunctions, in terms of ρ,

are computed using REDUCE 3.4 on a standard Pentium PC.

In figure 1 (Available from authour upon request) , 2D-PSLET results (solid curve) for the 1S-state compare excellently with the others in the weak magnetic field regime. In the strong field regime, 2D-PSLET results fall in between the perturbation (small dashes) [8] and direct numerical integration (solid curve connecting solid circles) [12], on which the series expansion results (solid circles on the solid curve) [9] are located. SLET and SLNT [3] results are presented by the long dashed curve. 2D-PSLET, SLET and SLNT results are, however, unique in their tendency to approach the strong and the weak magnetic field perturbation results [8]. The perturbation coupling constants were appropriately defined in these regimes. Likewise, we believe, it should be the tendency of the results of any approximation technique.

Figures 2-4 (Available from authour upon request) show that 2D-PSLET results (empty squares) are in exact agreements with those of the numerical solutions of the Schrödinger equation [10]. The best fit line of 2D-PSLET results exactly overlaps with that of the lowest two - point quasifractional approximation [10].

It should be mentioned that, for all states considered above, 2D-PSLET results for the energies, excluding the third - order correction, are exactly the same as those of SLET [3] and SLNT [3,5]. It is only because of the third - order correction, $E^{(2)}/\bar{l}^2$, that the 2D-PSLET results compared better, with the numerical results, than those of SLET and SLNT.

4 Conclusions and Remarks

In this work, the psuedoperturbative shifted - l expansion technique (PSLET) was introduced to find nodeless states of 2D Schrödinger equation with arbitrary cylindrically symmetric potentials. Exact energy eigenvalues and

eigenfunctions for 2D Coulomb and harmonic oscillator potentials were reproduced. Also, exact energy eigenvalues, compared to numerical ones [10], were obtained for the hybrid of the 2D Coulomb and oscillator potentials. The accuracy and rapid convergence of 2D-PSLET results are satisfactory and fascinating. The analytical results, tables, and figures clearly bear this out.

Finally, some observations concerning the attendant 2D-PSLET are in order. It is highly accurate and rapidly convergent, thus efficient with respect to computation time. Within the same procedure, it produces both eigenvalues and eigenfunctions. It puts no constraints on the coupling constants of the potential involved. It is to be understood as being an expansion through any existing quantum number in the centrifugal - like term of any Scrödinger - like equation, equation (1).

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Table 1: PSLET results for 1S-state energies (in effective Rydberg units) of the 2D hydrogenic levels in a magnetic field , where $EN_0 = \bar{l}^2 E^{(-2)}$, $EN_1 = EN_0 + E^{(0)}$, $EN_2 = EN_1 + E^{(1)}/\bar{l}$, and $EN_3 = EN_2 + E^{(2)}/\bar{l}^2$.

γ	EN_0	EN_1	EN_2	EN_3
0	-4.0	-4.0	-4.0	-4.0
1	-3.936821	-3.907949	-3.910053	-3.910538
2	-3.738814	-3.651631	-3.677083	-3.673240
3	-3.378562	-3.284620	-3.356622	-3.314095
4	-2.810985	-2.864030	-2.934645	-2.900042
5	-2.024429	-2.380567	-2.401652	-2.496601
6	-1.141379	-1.784237	-1.840713	-1.971307
7	-0.286914	-1.115087	-1.264520	-1.373630
8	0.525016	-0.421911	-0.660921	-0.761824
9	1.311535	0.279818	-0.032861	-0.139894
10	2.085744	0.987287	0.614648	0.494518
12	3.623940	2.419691	1.954631	1.802345
20	9.888892	8.387814	7.701685	7.438987
28	16.38730	14.65978	13.83081	13.48993
36	23.06128	21.13973	20.19640	19.79357
40	26.44867	24.43836	23.44415	23.01409

Table 2: PSLET results for 1S-state energies (in effective Rydberg units) of the 2D hydrogenic levels in a magnetic field , where $EN_0 = \bar{l}^2 E^{(-2)}$, $EN_1 = EN_0 + E^{(0)}$, $EN_2 = EN_1 + E^{(1)}/\bar{l}$, $EN_3 = EN_2 + E^{(2)}/\bar{l}^2$, and $\gamma^{'} = \gamma/(1+\gamma)$.

$\gamma^{'}$	EN_0	EN_1	EN_2	EN_3
0.0	-4.0	-4.0	-4.0	-4.0
0.1	-3.999228	-3.998843	-3.998843	-3.998830
0.2	-3.996091	-3.994148	-3.994156	-3.994162
0.3	-3.988498	-3.982838	-3.982914	-3.982940
0.4	-3.972089	-3.958670	-3.959106	-3.959233
0.5	-3.936821	-3.907949	-3.910053	-3.910542
0.6	-3.855896	-3.797597	-3.807128	-3.807128
0.7	-3.638726	-3.538277	-3.578849	-3.566445
0.8	-2.810985	-2.864030	-2.934645	-2.900042

Table 3: PSLET results for $2P^{-}$ - state energies (in effective Rydberg units) of the 2D hydrogenic levels in a magnetic field , where $EN_0 = \bar{l}^2 E^{(-2)}$, $EN_1 = EN_0 + E^{(0)}$, $EN_2 = EN_1 + E^{(1)}/\bar{l}$, $EN_3 = EN_2 + E^{(2)}/\bar{l}^2$, and $\gamma^{'} = \gamma/(1+\gamma)$.

$\gamma^{'}$	EN_0	EN_1	EN_2	EN_3
0.0	-4/9	-4/9	-4/9	-4/9
0.1	-0.525152	-0.523702	-0.523991	-0.523943
0.2	-0.556955	-0.562580	-0.562780	-0.562971
0.3	-0.542085	-0.561985	-0.563083	-0.563600
0.4	-0.481468	-0.514474	-0.518498	-0.519116
0.5	-0.355319	-0.400522	-0.408080	-0.409164
0.6	-0.113944	-0.172226	-0.183710	-0.185583
0.7	0.366646	0.292098	0.275860	0.272867
0.8	1.478312	1.379563	1.356515	1.351825

Table 4: PSLET results for $3D^{-}$ - state energies (in effective Rydberg units) of the 2D hydrogenic levels in a magnetic field , where $EN_0 = \bar{l}^2 E^{(-2)}$, $EN_1 = EN_0 + E^{(0)}$, $EN_2 = EN_1 + E^{(1)}/\bar{l}$, $EN_3 = EN_2 + E^{(2)}/\bar{l}^2$, and $\gamma^{'} = \gamma/(1+\gamma)$.

$\gamma^{'}$	EN_0	EN_1	EN_2	EN_3
0.0	-4/25	-4/25	-4/25	-4/25
0.1	-0.257041	-0.260346	-0.260476	-0.260508
0.2	-0.264976	-0.272340	-0.273106	-0.273176
0.3	-0.225665	-0.236222	-0.237588	-0.237735
0.4	-0.134531	-0.148204	-0.150156	-0.150392
0.5	0.031455	0.014366	0.011785	0.011447
0.6	0.325970	0.304776	0.301450	0.300990
0.7	0.881999	0.855342	0.851037	0.850414
0.8	2.115430	2.080314	2.074507	2.073635